Problem 2: Hilbert's Hotel Solution

We want to determine wether $n+1$ is prime. To just calculate whether $n+1$ is prime with the usual calculations, will become very time expensive when n becomes large. We are also given the number of ways we can order the n guests over the n rooms. First let us reason what this number is.

If $n = 1$, then we have only one guest and one room, so there is one way to order them.

If $n = 2$, then there are 2 guest and 2 rooms. For the first guest, we have 2 rooms to chose from. Once we have appointed the first guest to their room, the second guest automatically gets the other room. Thus we have 2 ways of appointing guests to the room.

If $n = 3$, then there are 3 guests and 3 rooms. For the first guest, we have 3 rooms to chose from. Once we have appointed the first guest to their room, we have still 2 rooms to choose from to appoint the second guest to. Once appointed the first two guests, we cannot choose anymore for the third guest and we have to appoint him/her to the only room left. Thus we have $3 \cdot 2 = 6$ ways of appointing the guests to the room.

This same method we can use for n guests and n rooms. For the first guest, we can choose n rooms for appointing. After appointing the first guest, for the second guest, we have $n-1$ rooms we can appoint the guest to. When we go on like this, we see that there are $n \cdot (n-1) \cdot ... \cdot 2 \cdot 1 = n!$ ways of appointing the n guests to the room.

We are given thus $n! \mod (n+1)$. Now we can note that if $n+1$ is not prime, we can write it as a product $n+1 = a \cdot b$ with $a, b < (n+1)$. Now as $a, b < (n+1)$, these numbers occur in $(n+1)$. Now, as long as we have $a \neq b$, this will result in $n! \mod (n+1) \equiv 0$. Now see that for every $n > 2$ not prime, except for $n = 4$, we can find such a, b. Note that when $n+1$ is prime, there do not exist such factors, so $n! \mod (n+1)$ is not equal to 0 (in fact, $n! \mod (n+1) \equiv n$, see `https://en.wikipedia.org/wiki/Wilson's_theorem`).